Discrete Random Variables: Expectation and Variance

1. X is a discrete random variable:

x	1	2	3	4
$\mathbb{P}(X = x)$	k	$2k$	$3k$	$4k$

   (a) Find the value of k. $\boxed{\frac{1}{10}}$

   (b) Find $\mathbb{E}(X)$. $\boxed{3}$

   (c) Find $\text{Var}(X)$. $\boxed{1}$

2. Consider the probability distribution:

x	1	2	3	4
$\mathbb{P}(X = x)$	a	b	$\frac{1}{3}$	$\frac{1}{4}$

   (a) By considering the probabilities, find an equation involving a and b. $\boxed{\frac{5}{12} = a + b}$

   (b) Given that $\mathbb{E}(X) = 2\frac{3}{4}$, find another equation involving a and b. $\boxed{\frac{3}{4} = a + 2b}$

   (c) Hence find a and b. $\boxed{a = \frac{1}{12}, b = \frac{1}{3}}$

   (d) Calculate $\text{Var}(X)$. $\boxed{\frac{41}{48}}$

3. A die is biased in such a way that the probability of scoring n ($n = 1$ to 6) is proportional to n. Find the mean (expected) score. $\boxed{\frac{13}{3}}$

4. In a game a player pays 3p each time he throws two unbiased dice. He receives back from the banker a number of pence equal to the difference of his score and 8. Find his expected winnings per go. $\boxed{-\frac{8}{9}\text{p}}$

5. A random number generator in a computer game produces values which can be modelled by the discrete random variable X with probability distribution given by

$$\mathbb{P}(X = x) = kx! \quad \text{for } x = 0, 1, 2, 3, 4$$
$$\mathbb{P}(X = x) = 0 \quad \text{otherwise.}$$

   where k is a constant.

   (a) Show that $k = \frac{1}{34}$ and illustrate the distribution with a sketch.

   (b) Find the expectation and variance of X. $\boxed{\frac{7}{2}, \frac{61}{68}}$

   Two independent values of X are generated. Let these values be X_1 and X_2.

   (c) Show that $\mathbb{P}(X_1 = X_2)$ is a little more than $\frac{1}{2}$. $\boxed{\frac{309}{578}}$

   (d) Given that $X_1 = X_2$, find the probability that X_1 and X_2 are each equal to 4. $\boxed{\frac{96}{103}}$

6. Two dice are rolled and the larger of the two scores is recorded (if they are the same, then the score of either dice is recorded). Calculate the probability of each possible score and calculate the expectation and variance of your recorded value. $\boxed{\frac{161}{36}, \frac{2555}{1296}}$

7. Two dice are rolled and the difference (always zero or a positive number) of the two scores is recorded (use D to represent this difference). Tabulate the possible scores with their probabilities. Calculate the expectation and variance of your recorded value. $\boxed{\frac{35}{18}, \frac{665}{324}}$

8. A bag contains 4 red balls and 6 black balls. Three balls are taken at once from the bag. Let X be the number of red balls in my selection. Create a probability table for the possible values of X. Calculate the expectation and variance of X. $\boxed{\frac{6}{5}, \frac{14}{25}}$

9. Three dice are rolled and the smallest value, X, recorded. Calculate

   (a) $\mathbb{E}(X)$. $\boxed{\frac{49}{24}}$

   (b) $\mathbb{E}(X^2)$. $\boxed{\frac{1183}{216}}$

   (c) $\mathbb{E}(X^2) - (\mathbb{E}(X))^2$. $\boxed{\frac{2261}{1728}}$

10. (a) Six tulip bulbs are planted in the ground. The probability of any one tulip bulb flowering is $\frac{2}{3}$, independent of any other tulip bulb flowering. Let B be the number of tulip bulbs that flower.

    i. Create a probability table for B.

    ii. Calculate the $\mathbb{E}(B)$. $\boxed{4}$

    iii. Calculate the $\mathrm{Var}(B)$. $\boxed{\frac{4}{3}}$

    (b) In the previous section could you have guessed the $\mathbb{E}(B)$ without doing any calculations? If 15 tulip bulbs are planted, each with a probability of $\frac{1}{5}$ of flowering, hazard a guess as to the Expected number of flowering tulips. Check with your teacher! $\boxed{3}$

11. A dice is rolled repeatedly until a six is obtained. Let T be the number of rolls until the six is obtained. For example if you rolled

$$2 \to 3 \to 3 \to 1 \to 2 \to 4 \to 6$$

then $T = 7$.

   (a) How many possible values could T take? $\boxed{\infty\text{-many}}$

   (b) Calculate $\mathbb{P}(T = 1)$. $\boxed{\frac{1}{6}}$

   (c) Calculate $\mathbb{P}(T = 4)$. $\boxed{\frac{125}{1296}}$

   (d) Calculate $\mathbb{E}(T)$. [You'll need some of the C2 material here.] $\boxed{6}$

   (e) Calculate $\mathrm{Var}(T)$. [*Ditto*] $\boxed{30}$

12. (Hard! Thanks to Dr MS) There are N doors in a corridor, each one initially open. Each time a person walks along the corridor he closes a random number of the doors – if there are k open doors, he is equally likely to close any number between 1 and k (note that he must close at least one door). This process stops when all of the doors are closed.

   (a) If $N = 1$ find the expected number of people to pass through the corridor during the process. (Obvious.) $\boxed{1}$

   (b) If $N = 2$ find the expected number of people to pass through the corridor during the process. $\boxed{\frac{3}{2}}$

   (c) If $N = 3$ find the expected number of people to pass through the corridor during the process. $\boxed{\frac{11}{6}}$

   (d) If $N = 4$ find the expected number of people to pass through the corridor during the process. $\boxed{\frac{25}{12}}$

   (e) If $N = n$ find the expected number of people to pass through the corridor during the process. $\boxed{1 + \frac{1}{2} + \cdots + \frac{1}{n}}$