

Really Nice Problem

If

$$x + \frac{1}{x} = 1,$$

then what is the value of

$$x^n + \frac{1}{x^n}?$$

Solution

Square both sides of the original equation:

$$\left(x + \frac{1}{x}\right)^2 = 1^2$$

so

$$x^2 + 2 + \frac{1}{x^2} = 1$$

so

$$x^2 + \frac{1}{x^2} = -1$$

multiply both sides by $x + \frac{1}{x}$ (which is just 1) to get

$$\left(x + \frac{1}{x}\right) \left(x^2 + \frac{1}{x^2}\right) = -1$$

so

$$x^3 + x + \frac{1}{x} + \frac{1}{x^3} = -1$$

so

$$x^3 + \frac{1}{x^3} = -2.$$

etc. Keep going by multiplying by $x + \frac{1}{x}$. You should see a pattern...