Determinants & Inverses of 3×3 Matrices

1. Find the value(s) of k for which each of the following determinants is zero:

   (a) $\det \begin{pmatrix} 4 & 3 & 4 \\ -3 & -2 & -1 \\ 2 & 1 & k \end{pmatrix}$.

   (b) $\det \begin{pmatrix} 5 & k & 8 \\ 2 & 6 & 3 \\ -1 & 4 & -2 \end{pmatrix}$.

   (c) $\det \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$.

2. Find the value of k for which each of the following sets of equations does not have a unique solution. [You need not solve them]:

   (a) $\begin{array}{rcrcrcl} 3x & + & 4y & + & 2z & = & 0 \\ 2x & + & 3y & + & 5z & = & 0 \\ 3x & + & 5y & + & kz & = & 0 \end{array}$.

   (b) $\begin{array}{rcrcrcl} 4x & - & 2y & + & 6z & = & 0 \\ x & + & ky & - & 3z & = & 0 \\ 3x & - & 5y & + & 9z & = & 0 \end{array}$.

   (c) $\begin{array}{rcrcrcl} kx & + & y & - & z & = & 0 \\ x & + & 3y & - & 2z & = & 0 \\ 5x & + & & & z & = & 0 \end{array}$.

3. Expand and simplify the following determinants:

   (a) $\det \begin{pmatrix} 1 & 1 & 1 \\ \cos\theta & \cos^2\theta & 1 \\ \sin\theta & \sin^2\theta & 1 \end{pmatrix}$.

   (b) $\det \begin{pmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix}$.

   (c) $\det \begin{pmatrix} \cos\theta & 1 & 1 \\ \sin\theta & 1 & 1 \\ 1 & \sin\theta & \cos\theta \end{pmatrix}$.

   (d) $\det \begin{pmatrix} 1 & n & n^2 \\ n & n^2 & n^3 \\ 1/n & 1/n^2 & 1/n^3 \end{pmatrix}$.

4. Solve the equation
$$\det \begin{pmatrix} 1 & 1 & 1 \\ x & x+1 & x-1 \\ x-1 & 2x & x+1 \end{pmatrix} = 0.$$

5. Find the inverse of the following matrices:

   (a) $\begin{pmatrix} 1 & -2 & -3 \\ 2 & -1 & -4 \\ 3 & -3 & -5 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 5 \\ 3 & -2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 3 & -4 & 5 \end{pmatrix}$

6. By writing the following sets of linear equations in matrix form, solve the system of equations by inverting the resulting matrix:

(a)
$$\begin{array}{rcrcrcr} & x & + & y & + & z & = & 4 \\ & 2x & - & y & - & z & = & -1 \\ & x & + & 2y & + & 3z & = & 9 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} & x & - & y & - & z & = & 3 \\ & x & + & 2y & - & 3z & = & -5 \\ & 2x & + & 2y & + & z & = & 1 \end{array}$$

(c)
$$\begin{array}{rcrcrcr} & 2x & + & y & - & z & = & 3 \\ & 4x & + & y & - & 3z & = & 2 \\ & & & 2y & - & 5z & = & 1 \end{array}$$

7. It is given that the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & a \\ 9 & -5 & -7 \end{pmatrix}$ is non-singular.

(a) Find the set of possible values of a.

(b) Find the inverse of $\mathbf{M}$.

(c) Solve the equations
$$\begin{array}{rcrcrcr} 2x & - & & & z & = & 1 \\ & & 4y & + & az & = & -3 \\ 9x & - & 5y & - & 7z & = & 2 \end{array}$$
for x, y and z in terms of a.