Differentiating Natural Logarithms & Exponentials

1. Given that $y = \ln(1+x) - \frac{x}{1+x}$, find $\frac{dy}{dx}$ and show that y is positive for all positive values of x.

2. Differentiate e^{2x} and $\frac{e^{2x}}{1+e^x}$ with respect to x.

   Hence find the coordinates of the stationary point and determine its nature. Sketch the curve.

3. Find the co-ordinates of the stationary points of the curve $y = x^2e^{-x}$, and determine its nature. Hence sketch $y = x^2e^{-x}$.

   Explain, from your graph, why, if $0 < k < \frac{4}{e^2}$, the equation $ke^x = x^2$ has three real roots.

4. Find the local maximum and minimum values of $y = xe^{-2x^2}$.

   Find where the curve crosses the axes.

   Sketch the curve (make it reasonably large).

   Find also the points of inflexion of the curve (and they do not occur where $\frac{dy}{dx} = 0$).

   Identify these inflexions on your sketch.

5. Calculate the co-ordinates of the stationary point of the graph of $\frac{\ln x}{x}$ and prove whether it is a maximum, minimum or point of inflexion.

   Sketch the graph of the function for the domain $x > 0$.

   Given that $1 < a < b$, and $\frac{\ln a}{a} = \frac{\ln b}{b}$, show that there is a number k such that $a < k < b$ whatever pair of a and b are chosen. Find this value of k.

   With the help of your sketch, or otherwise, find all the pairs of positive integers (x, y) such that

   $$x^y = y^x.$$

6. (a) If $y = e^{-x}\ln(1+x)$, show that

   $$\frac{dy}{dx} = e^{-x}\left(\frac{1}{1+x} - \ln(1+x)\right)$$

   and hence that

   $$(1+x)\frac{d^2y}{dx^2} + (2x+3)\frac{dy}{dx} + (x+2)y = 0.$$

   (b) Using the same set of axes, sketch the graphs of $y = \ln(1+x)$ and $y = \frac{1}{1+x}$. Hence show that the function $y = e^{-x}\ln(1+x)$ has exactly one stationary point, which occurs for a positive value of x and, from your sketch or by using (a), identify its nature.

   (c) Sketch the graph of $y = e^{-x}\ln(1+x)$.

7. Find the stationary points of $y = \ln(x^3 - 3x + 3)$.

   Find where the curve meets the axes.

   Sketch the curve.

8. Differentiate x^ne^{-x}.

   Denoting this function by $f_n(x)$, prove that $f_n(x)$ attains its maximum value in the range $x \geqslant 0$ when $x = n$.

(a) By considering $f_n(n)$ and $f_n(n+1)$ prove that $\left(1+\frac{1}{n}\right)^n < e$.

By considering $f_{n+1}(n)$ and $f_{n+1}(n+1)$ prove that $\left(1+\frac{1}{n}\right)^{n+1} > e$, and thus

$$\left(1+\frac{1}{n}\right)^n > \frac{e}{\left(1+\frac{1}{n}\right)}.$$

Hence show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

(b) Similarly show that $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}$.

(c) Using (a) and (b), suggest a value for $\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n$, and confirm it with numerical approximations on your calculator.

(d) Find the following:

i. $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n$;

ii. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\sqrt{n}}$;

iii. $\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^n$;

iv. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.