C2 Logarithms

Patrons are reminded that if you see a log with no base then it means $\log_{10}$. For example $\log 7 \equiv \log_{10} 7$. You are also reminded that if you have a number which is not a log (3, say) and you want to write it in terms of a logarithm then this is how to do it:

$$3 = 3 \times 1 = 3 \times \log_a a = \log_a (a^3).$$

1. Write down (without a calculator) the value of the following logarithms:

   (a) $\log_2 8$. $\boxed{3}$

   (b) $\log_{10} 100$. $\boxed{2}$

   (c) $\log_3 1$. $\boxed{0}$

   (d) $\log_9 3$. $\boxed{\frac{1}{2}}$

   (e) $\log_{10}\left(\frac{1}{100}\right)$. $\boxed{-2}$

   (f) $\log_2\left(\frac{1}{16}\right)$. $\boxed{-4}$

   (g) $\log_a (a^6)$. $\boxed{6}$

   (h) $\log_{\sqrt{a}} (a^2)$. $\boxed{4}$

2. State (without a calculator) two *consecutive* integers that the following logarithms lie between:

   (a) $\log_{10} 20$. $\boxed{\text{2 and 3}}$

   (b) $\log_5 300$. $\boxed{\text{3 and 4}}$

   (c) $\log_3 2$. $\boxed{\text{1 and 2}}$

   (d) $\log_{10} 0.2$. $\boxed{-1 \text{ and } 0}$

3. Express the following as a single logarithm:

   (a) $\log_a x + \log_a (x^2)$. $\boxed{\log_a x^3}$

   (b) $\log_2(x^3) - \log_2(x^2)$. $\boxed{\log_2 x}$

   (c) $\log_c a + \log_c (ab)$. $\boxed{\log_c a^2 b}$

   (d) $2 \log x + 3 \log y$. $\boxed{\log x^2 y^3}$

   (e) $2 \log_5 x - \log_5 y + \log_5 z$. $\boxed{\log_5\left(\frac{x^2 z}{y}\right)}$

   (f) $2\log_7(s^2) - 3\log_7(s^3) + 5\log_7 t$. $\boxed{\log_7\left(\frac{t^5}{s^5}\right)}$

   (g) $\log a - 3 \log b - 7 \log c + 1$. $\boxed{\log\left(\frac{10a}{b^3 c^7}\right)}$

   (h) $2 \log_a p + \log_a q - 7 \log_a r - 3$. $\boxed{\log_a\left(\frac{p^2 q}{a^3 r^7}\right)}$

4. Solve the following equations (if there is a logarithm in brackets after the question, please use logarithms to *that* base to solve the problem, even if it is unnatural to use that base). Give all answers to three significant figures, where appropriate.

   (a) $2^x = 5$. $(\log_{10})$ $\boxed{x = 2.32}$

   (b) $8^x = 3$. $(\log_8)$ $\boxed{x = 0.528}$

   (c) $3^{2x} = 11$. $(\log_3)$ $\boxed{x = 1.09}$

   (d) $5^{3x-4} = 100$. $(\log_{10})$ $\boxed{x = 2.29}$

   (e) $17 = 13^{x-4}$. $(\log_5)$ $\boxed{x = 5.10}$

   (f) $2^x 2^{x+1} = 10$. $(\log_2)$ $\boxed{x = 1.16}$

   (g) $5 = 7 \times 2^{x+1}$. $(\log_{10})$ $\boxed{x = -1.49}$

   (h) $3 \times 2^{2-3x} = 13$. $(\log_3)$ $\boxed{x = -0.0385}$

   (i) $a \times b^{x+c} = d$. $(\log_d)$ $\boxed{x = \frac{1 - \log_d a - c\log_d b}{\log_d b}}$

   (j) $11 \times 9^x = 13$. $(\log_3)$ $\boxed{x = 0.0760}$

   (k) $3^x = 2^{x+1}$. $(\log_5)$ $\boxed{x = 1.71}$

   (l) $3^{x+1} = 4^{2x-1}$. $(\log_4)$ $\boxed{x = 1.48}$

   (m) $2 \times 3^x = 5^{1-x}$. $(\log_{10})$ $\boxed{x = 0.338}$

   (n) $7 \times 2^{2x+1} = 6 \times 11^{x+1}$. $(\log_2)$ $\boxed{x = -1.53}$

   (o) $a \times b^{cx+d} = e \times f^{gx+k}$. $(\log_z)$ $\boxed{x = \frac{\log_z e - \log_z a + k\log_z f - d\log_z b}{c\log_z b - g\log_z f}}$

5. Solve the following equations (you may need the factor theorem for the later problems):

(a) $\log_2 x - \log_2(x-1) = 3$. [$x = \frac{8}{7}$]

(b) $\log_3(x+2) + \log_3 x = 1$. [$x = 1$ (only)]

(c) $\log_3(2x) - \log_3(1-x) = 2$. [$x = \frac{9}{11}$]

(d) $2 = \log_2(2x) + \log_2(x-1)$. [$x = 2$ (only)]

(e) $\log_2 x + \log_2(2x+1) = 6$. [$x = \frac{\sqrt{513}-1}{4}$ (only)]

(f) $\log_2(x-1) = 4 + \log_2(2x+3)$. [No solns]

(g) $2\log_5 x + \log_5 x = 3$. [$x = 5$]

(h) $2\log_2(x+3) + \log_2 x - \log_2(4x+2) = 1$. [$x = \frac{\sqrt{41}-5}{2}$ (only)]

6. Solve the following equations:

(a) $2^{2x} + 15 = 8 \times 2^x$. [$x = \log_2 3$ or $x = \log_2 5$]

(b) $8 \times 3^x = 3^{2x} + 7$. [$x = 0$ or $x = \log_3 7$]

(c) $5^{2x} = 16 - 6 \times 5^x$. [$x = \log_5 2$ (only)]

(d) $4 \times 7^x + 7^{2x} + 3 = 0$. [No solns]

(e) $3^x + 1 = \frac{72}{3^x}$. [$x = \log_3 8$ (only)]

(f) $3 \times 2^{2x} + 5 = 16 \times 2^x$. [$x = \log_2(\frac{1}{3})$ or $x = \log_2 5$]

(g) $4 \times 3^{2x} = 35 + 4 \times 3^x$. [$x = \log_3(\frac{7}{2})$ (only)]

(h) $2^{2x} + 35 = 3 \times 2^{x+2}$. [$x = \log_2 5$ or $x = \log_2 7$]

7. Given that $x = \log_a p$ and $y = \log_a q$, write the following in terms of x and y

(a) $\log_a\left(p^2 q\right)$. [$2x + y$]

(b) $\log_a\left(\frac{q}{\sqrt{p}}\right)$. [$y - \frac{x}{2}$]

(c) $\log_a\left(p^2 q\right) - 2\log_a\left(\frac{q}{p}\right)$. [$4x - y$]

(d) $\log_a\left(\sqrt{pq^3}\right) - \frac{1}{2}\log_a(qp)$. [$y$]

(e) $\log_p a$. [$\frac{1}{x}$]

(f) $\log_p q$. [$\frac{y}{x}$]

8. Find the intersection of the curves $y = \log_2 x + 3$ and $y = \log_2(x+3)$. [$(x, y) = (\frac{3}{7}, \log_2 24 - \log_2 7)$]

9. (a) Show that if $\log a + \log c = 2\log b$ then a, b and c are in geometric progression.
   (b) Show that if $\log x + \log z = 3\log y$ then x, y^2 and yz are in geometric progression.

10. The definition of a logarithm is given by $a = b^c \quad \Leftrightarrow \quad c = \log_b a$.

(a) Take $a = b^c$ and this time take logs to the base c of both sides of the equation and hence prove that
$$\log_c b \times \log_b a = \log_c a.$$

(b) Hence or otherwise calculate to 4 significant figures $\log_3 5$.

(c) Deduce $\log_3 25$ and $\log_3\left(\frac{\sqrt{5}}{3}\right)$.

11. Taking the same scale on the x and y-axes, draw a separate sketch for each of the following:

(a) $y = \log_2 x$.

(b) $y = \log_2(-x)$.

(c) $y = \log_2(x+3)$.

State how $y = \log_2 x$ can be transformed into each of the other two.

12. (a) Write each of 169 and 243 as a product of prime numbers.
    (b) Write $x = \log_3 169$ in index form.
    (c) Evaluate $\log_3 169 \times \log_{13} 243$ without using a calculator.

13. A firm is testing two types of scrubbing brush by using a machine that keeps the brushes in continuous action.

   (a) The first brush starts with 2000 bristles and the number of bristles, n, left after t days is known to follow the rule
   $$n = 2000 \times 2^{-t/100}.$$
   Find the number of bristles left after 10 days.

   (b) The second brush starts with 1450 bristles and follows the rule
   $$n = A \times 3^{-t/P}$$
   where A and p are constants. After 10 days it is found to have 1373 bristles. Write down the value of A and calculate the value of p to the nearest 10.

14. It is often easy to prove that many logarithms are irrational numbers, and a method of proof may be *reductio ad absurdam* (proof by contradiction).

   For example, consider $\log_m n$. Suppose that m and n are natural numbers (i.e. numbers from the set $\{1,2,3,4,\ldots\}$) and, first, that one is odd and the other even. Using *reductio ad absurdam*, prove that $\log_m n$ is irrational.