C2 Integration Definite

1. Sketch the graph of $y = 4x^2 - 9$, and find the area bounded by this curve and the x-axis.

2. Find the areas bounded by $y = 4(x^2 - 4)(3 - x)$ and the x-axis (i) for which $y > 0$; (ii) for which $y < 0$. Write down the value of
$$\int_{-2}^{3} 4(x^2 - 4)(3 - x)\,dx.$$

3. Evaluate $\int_1^3 (4 + 3x - x^3)\,dx$. What can you deduce from your results about the graph of $y = 4 + 3x - x^3$ between $x = 1$ and $x = 3$?

4. Evaluate $\int_0^2 (x^2 + 1)\,dx$. Explain, with the aid of a sketch, how you could deduce the value of
$$\int_{-2}^{2} (x^2 + 1)\,dx.$$
Show, preferably by means of another sketch, that
$$\int_{-2}^{2} x(x^2 + 1)\,dx.$$

5. Sketch the curve $y = x(x - 2)$. Find an expression for
$$F(t) \equiv \int_{-1}^{t} x(x - 2)\,dx$$
and sketch the curve $y = F(t)$ for $-1 \leqslant x \leqslant 3$.
Find the maximum and minimum values of $F(t)$, and the values of t for which these occur. How do these relate to the first graph?

6. (a) Use the result
$$\int_a^b f(x)\,dx = F(b) - F(a),$$
where $F'(x) \equiv f(x)$, to show that
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$
and interpret this in terms of areas when $a < b < c$.

   (b) Show that putting $a = b$ in (a) suggests that
$$\int_a^a f(x)\,dx = 0,$$
and explain how this fits with the interpretation of a definite integral as an area.

   (c) Show that putting $c = a$ in (a) suggests that
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$
[The results in (b) and (c) are used as the definitions of $\int_a^b f(x)\,dx$ when $a \geqslant b$.]

7. Calculate the coordinates of the points where the curve $y = x^2$ crosses the line $y = 3x + 10$ and show them in a sketch of a line and the curve. On your diagram shade the area which satisfies both $y \geqslant x^2$ and $y \leqslant 3x + 10$. Calculate this area.

8. Sketch the curves $y = x^2$ and $y = 8 - x^2$, and show that they intersect at the points $(2,4)$ and $(-2,4)$. Calculate the area enclosed by the two curves.

9. Find the area enclosed by the curves $y = x^2$ and $y = x^3$.

10. Sketch the curves
$$y = \frac{4}{x^2},\ y = x(x+3) \text{ and } y = x - \frac{x^2}{4}.$$
for $0 \leqslant x \leqslant 2$. You *should* find they cross at $(0,0)$, $(1,4)$ and $(2,1)$ and that the three curves enclose an area between these points. Find the enclosed area.

11. Show that if $n \geqslant 1$, the area enclosed by the curves $y = x^n$ and $y = x^{1/n}$ is $\dfrac{n-1}{n+1}$.

12. Sketch the curve $y = (x+2)^2$, and shade the region defined by the inequalities $x \geqslant 0$, $y \geqslant (x+2)^2$, $y \leqslant 9$. Calculate the area of this region.

13. Draw a rough sketch of the curve $y^2 = 8x + 8$. Calculate the area enclosed by the curve and the line $x = 1$.