C1 Discriminant

Given a quadratic expression (ax^2+bx+c) or equation $(ax^2+bx+c=0)$ the 'discriminant' is defined b^2-4ac. In the context of an equation we have:
$b^2-4ac>0 \Rightarrow$ Equation has two distinct roots.
$b^2-4ac=0 \Rightarrow$ Equation has one repeated root.
$b^2-4ac<0 \Rightarrow$ Equation has no real roots (but two complex roots in FP1).
If the discriminant is zero this often hints at a tangent to a circle or a quadratic curve.

1. Calculate the discriminant for the following quadratic equations.
   (a) $3x^2-2x-5=0$. [64]
   (b) $x^2=3x+1$. [13]
   (c) $-x^2+6x=2$. [28]
   (d) $2x^2-3x+1=3x^2+5x$. [68]
   (e) $kx^2+k=x$. [$1-4k^2$]
   (f) $kx^2+2kx=k$. [$8k^2$]
   (g) $x^2+ax=bx+1$. [$a^2-2ab+b^2+4$]
   (h) $ax^2+bx+c=bx^2+cx+a$. [$b^2+c^2+2bc-4ac-4ab+4a^2$]
2. How many solutions does $4x^2-3x+2=0$ have? [0]
3. How many solutions does $5x+3x^2=20-x$ have? [2]
4. How many solutions does $x^2+kx-5=0$ have? [2]
5. Find the value(s) of k for which $kx^2+5x+1=0$ has exactly one solution. [$k=\frac{25}{4}$]
6. Find the value(s) of k for which $x^2+1=kx$ has two distinct solutions. [$k>2$ or $k<-2$]
7. Find the value(s) of k for which $x^2+kx=k$ has equal roots. [$k=0$ or $k=-4$]
8. Find the value(s) of k for which $kx^2-kx+5=0$ has no real solutions. [$0<k<20$]
9. Find the value(s) of k for which $kx^2=x+1$ has two distinct solutions. [$k>-\frac{1}{4}$]
10. Find the value(s) of k for which $kx^2+2=kx$ has no real solutions. [$0<k<8$]
11. Find the value(s) of k for which $x^2+kx=x-25$ has exactly one solution. [$k=11$ or $k=-9$]
12. Find the value(s) of k for which $2x^2+kx+1=2x$ has exactly one solution. [$k=2\pm2\sqrt{2}$]
13. Find the value(s) of t for which $tx^2+2tx+t=x$ has no real solutions. [$t>\frac{1}{4}$]
14. Find the value(s) of k for which $ax^2-kx+a=0$ has two distinct solutions. [$k>2a$ or $k<-2a$]
15. Find the value(s) of c for which $y=4x+c$ lies tangent to $y=x^2+6x+1$. [$c=0$]
16. Find the value(s) of m for which $y=mx-2$ lies tangent to $y=x^2$. [$m=\pm2\sqrt{2}$]
17. Find the value(s) of m for which $y=mx-3$ lies tangent to $y=x^2+1$. [$m=\pm4$]
18. Find the value(s) of c for which $y=x+c$ lies tangent to the circle $x^2+y^2=4$. [$c=\pm2\sqrt{2}$]
19. Find the value(s) of c for which $y=2x+c$ lies tangent to $x^2+y^2=9$. [$c=\pm3\sqrt{5}$]

20. Find the value(s) of m for which $y = mx - 3$ lies tangent to the circle $x^2 + (y-1)^2 = 1$.

$m = \pm\sqrt{15}$

21. In this question a and b are distinct, non-zero real numbers, and c is a real number.

   (a) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

   has two distinct real solutions.

   (b) Show that the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

   has exactly one real solution if $c^2 = -\frac{4ab}{(a-b)^2}$. Show that this condition can be written $c^2 = 1 - \left(\frac{a+b}{a-b}\right)^2$ and deduce that it can only hold if $0 < c^2 \leqslant 1$. [STEP I 2005]