Core 1 Fun

1. Given that $x = y^{-\frac{2}{3}}$ and $y = \frac{8}{\sqrt{z}}$, express x as a power of z.

2. You are given that $f(x) = x^3 + 3x - 14$.

   (a) Find $f'(x)$ and deduce that the minimum gradient of the curve $y = f(x)$ is 3.

   (b) Deduce that the equation $f(x) = 0$ has only one real solution.

   (c) It is clear here by inspection that the solution to the equation is $x = 2$. Here we are going to use Vieta's method to solve it.

      i. Use the substitution $x = u - \frac{1}{u}$ in the equation $x^3 + 3x - 14 = 0$ to show that
      $$u^6 - 14u^3 - 1 = 0.$$

      ii. Solve this disguised quadratic to find two solutions for x^3 in the form $a + b\sqrt{r}$ where a, b and r are integers and r is square free

      iii. Expand $(1 + \sqrt{2})^3$ and hence write down the expansion of $(1 - \sqrt{2})^3$

      iv. Deduce that the two real values of u and show that for both of these, $x = 2$.

3. The Circle C has centre the origin and radius $10\sqrt{2}$.

   (a) Show that the point $P(2, 14)$ lies on C and find the equation of the tangent L to the circle C at P, leaving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

   (b) Show that the point $R(-5, 15)$ lies on the line L and calculate the exact length of the segment PR.

   (c) The point Q is the point, other than P, that lies on the circle C and whose tangent passes through R

      i. Write down the length QR.

      ii. By considering the intersection of C with a circle centre R which passes through Q, find the coordinates of Q.

4. (a) Solve the inequality
   $$12k^2 - 4k - 1 \leqslant 0.$$

   (b) Find the range of values of k such that the equation
   $$kx^2 - x + (3k - 1) = 0$$
   has at least one real solution. Deduce that
   $$-\frac{1}{6} \leqslant \frac{x+1}{x^2+3} \leqslant \frac{1}{2}$$

   (c) Use the same method to find the maximum and minimum values of
   $$\frac{x^2 - x + 1}{x^2 + x + 1}.$$