Serial Number 32

THE COLLEGES OF OXFORD UNIVERSITY

Entrance Examination in Mathematics

MATHEMATICS I

14 November 1994 Afternoon

Time allowed: 3 hours

Answers to each of Sections A, B and C must be attached to separate cover sheets and handed in separately. If no questions are attempted in any one section the cover sheet should still be handed in. Each cover sheet should be clearly labelled A, B or C.

All candidates must attempt Question 1 which carries twice the mark for any other question. There is no restriction on the number of questions any candidate may attempt but only Question 1 and the best three solutions to Questions 2-11 will contribute to the total mark for this paper.

The use of calculators is allowed, but, unless otherwise stated, exact answers should be given.

SECTION A

A1. (i) Differentiate $\sin(\cos x^2)$ with respect to x.

(ii) Express $\dfrac{1}{(x-1)(x^2+3)}$ in partial fractions.

(iii) Evaluate $\displaystyle\int_1^e (1+x)\ln x \, dx$, expressing the answer in terms of e.

(iv) Evaluate $\dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \cdots + \dfrac{1}{4n^2-1}$.

(v) Let a, b and c be positive integers greater than 1. Show that, if $x = \log_a b$, $y = \log_b c$ and $z = \log_c a$, then $xyz = 1$.

(vi) Find the greatest and least values of $x^3 - 3x + 1$ in the interval $-3/2 \le x \le 5/2$.

(vii) Let $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. Find vectors $\mathbf{p}$ and $\mathbf{q}$ with $\mathbf{p}$ parallel to $\mathbf{a}$ and $\mathbf{q}$ perpendicular to $\mathbf{a}$ such that $\mathbf{b} = \mathbf{p} + \mathbf{q}$.

(viii) Show that $x^2 + y^2 - 2x + 4y - 4 = 0$ is the equation of a circle, and find its centre and radius.

(ix) Give a formula for the distance between the points $(3, 0)$ and (t, t^2). Hence determine the minimum distance between the point $(3, 0)$ and the curve given by the parametric equations $x = t$, $y = t^2$.

(x) Sketch the graph of $y = \dfrac{ax^2}{(x^2+1)}$, where $a > 0$, and give the equations of any asymptotes.

SECTION B

B2. (a) Show that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, where $\binom{m}{k} = \dfrac{m!}{k!(m-k)!}$.

(b) Prove that, when n is a positive integer,

$$(a+x)^n = a^n + na^{n-1}x + \ldots + \binom{n}{r} a^{n-r}x^r + \ldots + x^n.$$

(c) Determine all values of a and b such that, when $(x+a)^3(x-b)^6$ is expanded as a sum of powers of x, the coefficient of x^8 is zero and the coefficient of x^7 is -9.

B3. Let α, β and γ be the solutions of the cubic equation $x^3 + Ax^2 + Bx + C = 0$.

(a) Show that

$$\begin{aligned} \alpha+\beta+\gamma &= -A, \\ \alpha\beta+\beta\gamma+\gamma\alpha &= B, \\ \alpha\beta\gamma &= -C. \end{aligned}$$

(b) Given that $A = -1, B = 5$ and $C = -3$, find the cubic equation whose solutions are $\beta+\gamma, \gamma+\alpha$ and $\alpha+\beta$.

B4. (a) Find all solutions of the equation $16x^4 - 8x^2 = -1$.

(b) Show that

$$\sin 3\theta = 4\cos^2\theta\sin\theta - \sin\theta$$

and that

$$\sin 5\theta = 16\cos^4\theta\sin\theta - 12\cos^2\theta\sin\theta + \sin\theta.$$

(c) Find the general solution of the equation

$$\sin 5\theta + \sin 3\theta + \sin\theta = 0.$$

B5. Find all solutions of the equations

$$\begin{aligned} 5x - 4y + kz &= 6 \\ kx - 3y + 6z &= 7 \\ 4x + 2y - 2z &= 2 \end{aligned}$$

(a) when $k = 0$, and (b) when $k = 7$.

B6. (a) Prove that $\left(\dfrac{1+i\sqrt{3}}{2}\right)^3 = -1$.

(b) Find the quadratic equation whose solutions are $\left(\dfrac{1+i\sqrt{3}}{2}\right)$ and $\left(\dfrac{1-i\sqrt{3}}{2}\right)$.

(c) Let $z = \left(\dfrac{1+i\sqrt{3}}{2}\right)$. Show that, for all values of the integers p, q and r,

$$(p - qz + rz^2)(p - rz + qz^2)$$

is an integer.

SECTION C

C7. Suppose that, for all $x, u(-x) = u(x)$ and $v(-x) = -v(x)$.
Explain why

$$\int_{-a}^{a} u(x)v(x)\mathrm{d}x = 0$$

for any finite value of a.

By considering the integral $\int_{-\pi/2}^{\pi/2} t\sin^3(\frac{1}{2}\pi + t)\mathrm{d}t$, or otherwise, show that

$$\int_0^{\pi} x\sin^3 x dx = \frac{\pi}{2}\int_0^{\pi} \sin^3 x \mathrm{d}x.$$

The function $F(x)$ is such that $F'(x) = \sin^3 x$. By considering the integral $\int_0^{\pi} xF'(x)\mathrm{d}x$, or otherwise, show that

$$\frac{2}{\pi}\int_0^{\pi} F(x)\mathrm{d}x = F(\pi) + F(0).$$

C8. It is known that, if the functions $f(x)$ and $g(x)$ satisfy $f(a) = g(a)$ and, for $a \le x \le b$,

$$f'(x) \ge g'(x),$$

then $f(b) \ge g(b)$, and the equality $f(b) = g(b)$ holds only if $f'(x) = g'(x)$ for $a \le x \le b$. By means of this result, ***which you are NOT asked to prove***, or otherwise, show that, for all $x > 0$,

$$x > \tan^{-1} x > \frac{x}{(1+x^2)}.$$

Show also that, for all $x \neq 0$,

$$x^2 > x\tan^{-1} x > \ln(1+x^2).$$

[Observe that the function $\tan^{-1} x$ is the same as $\arctan x$ and its derivative is $\frac{1}{1+x^2}$.]

Turn Over

C9. Calculate the derivative of the function $f(x) = \ln(x)/x$, for $x > 0$.
Show that $f'(x) < 0$ when $x > \mathrm{e}$.
Find any turning points on the graph of $f(x)$, and sketch the graph for $x > 0$.
[You may assume that $\ln(x)/x \to 0$ as $x \to \infty$.]
From your graph determine the range of values of c for which the equation $\ln(x)/x = c$ has

a) no solution,
b) exactly one solution,
c) more than one solution.

Using these results, or otherwise, show that, if the positive integers a and b are such that $a^b = b^a$ and $b > a$, then $a = 2$. What is the value of b?

C10. For positive integers m and n and real constants a and b with $a < b$, let

$$I(m,n) = \int_a^b (b-x)^m (x-a)^n \mathrm{d}x.$$

By integrating by parts, show that, for $m \geq 1$,

$$I(m,n) = \frac{m}{n+1} I(m-1, n+1).$$

Deduce that

$$I(m,n) = \frac{m!n!}{(m+n+1)!}(b-a)^{m+n+1}.$$

Find $\int_{-1}^{1} (1-x^2)^n \mathrm{d}x$.

rely C11. (i) Show that the substitution $x = \sin t$ transforms the differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad (-1 < x < 1)$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y = 0.$$

Find y in terms of x, given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.
(ii) By putting $y = vx$, transform the differential equation

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y^2}{x^2} \quad (x > 0)$$

into one connecting v and x and hence find y in terms of x. Unfinished!