MARCH

Serial Number 32

# THE COLLEGES OF OXFORD UNIVERSITY

## Entrance Examination in Mathematics

## MATHEMATICS I

15 November 1993 Afternoon

Time allowed: 3 hours

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Answers to each of Sections A, B and C must be attached to separate cover sheets and handed in separately. If no questions are attempted in any one section the cover sheet should still be handed in. Each cover sheet should be clearly labelled A, B or C.

All candidates must attempt Question 1 which carries twice the mark for any other question. There is no restriction on the number of questions any candidate may attempt but only Question 1 and the best three solutions to Questions 2-11 will contribute to the total mark for this paper.

*The use of calculators is allowed, but, unless otherwise stated, exact answers should be given.*

Turn Over

## SECTION A

A1 (i) Differentiate $\exp\{\exp(\sin x^5)\}$ with respect to $x$.

(ii) For $n$ a nonnegative integer let

$$I_n = \int_0^1 \frac{x^n}{1+x^2} dx.$$

Prove that, for $n \geq 2$,

$$I_n + I_{n-2} = \frac{1}{n-1}.$$

Hence find $I_5$.

(iii) Find the finite area between the curves $y = 4x^2$ and $x = 4y^2$.

(iv) Evaluate

$$\int \frac{\sin 2x}{\cos^n x} dx$$

when $n > 2$.

(v) Find constants $a$, $b$ such that

$$5 - x^2 - 4x \equiv a - (x+b)^2.$$

Hence evaluate

$$\int_{-2}^{1} \frac{dx}{\sqrt{5 - x^2 - 4x}}.$$

(vi) Sketch the curve $y = (x^2 - 3)e^x$ and find the values of $y$ at the turning points.

(vii) Find the sum of the first $n$ positive even integers and the sum of the first $n$ positive odd integers.

(viii) Let $\mathbf{a}$ and $\mathbf{b}$ be the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}.$$

Find $t$ such that the vector $\mathbf{a} + t\mathbf{b}$ is perpendicular to $\mathbf{a}$. Find a non-zero vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(ix) Express

$$\frac{2x+2}{2x^2 - 5x - 3}$$

in partial fractions.

(x) Find the values of $x$ such that $0 \leq x \leq 2\pi$ and $\sin 2x = |\sin x|$.

## SECTION B

B2. Consider the following set of linear equations:

$$\begin{aligned} x \quad -y \quad\quad\;\; &= 2 \\ 4x + 3y - z &= 3 \\ 8x + 13y - 3z &= a \end{aligned}$$

(a) Show that the equations have no solutions unless $a = 1$. When $a = 1$ find the general solution.

(b) Suppose that equation $5x - 3y = 4$ is added to the set above and $a = 1$. Do the equations then have a solution? If so, find it.

B3. Suppose that $r$, $s$, $\theta$, and $\phi$ are real numbers, and that $r > 0$ and $s > 0$. Let

$$A(r,\theta) = \begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A(1,0).$$

(a) Show that $A(r,\theta)A(s,\phi) = A(rs,\theta+\phi)$.

(b) Find in terms of $r$ and $\theta$ the values of $s$ and $\phi$ such that $A(r,\theta)A(s,\phi) = I$.

Let $n$ be a positive integer.

(c) Find all the values of $r$ and $\theta$ for which $[A(r,\theta)]^n = I$.

(d) Find all the values of $r$ and $\theta$ for which $[A(r,\theta)]^n = A(s,\phi)$.

B4. Let $\alpha$, $\beta$ and $\gamma$ be the roots of the cubic equation

$$x^3 + Ax^2 + Bx + C = 0.$$

(a) Show that

$$\begin{aligned} \alpha + \beta + \gamma &= -A, \\ \alpha\beta + \beta\gamma + \gamma\alpha &= B, \\ \alpha\beta\gamma &= -C. \end{aligned}$$

(b) Find expressions for $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$ in terms of $A$, $B$ and $C$.

(c) Find $A$ when $\alpha = -\beta$, $B = -4$, and $C = 16$.

B5. Let $n$ be a positive integer. Write down an expression for the expansion of $(1+x)^n$ as a polynomial in $x$ with coefficients $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $r = 0, 1, \ldots, n$.

(a) (i) Show that

$$\binom{n}{r} = \binom{n}{n-r};$$

(ii) show that

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

(b) By considering $(1+x)^{n-1}(1+x)$, or otherwise, show that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

(c) By considering the derivative of $(1+x)^n$, or otherwise, show that

$$\sum_{r=0}^{n} r \binom{n}{r} = n2^{n-1},$$

and find

$$\sum_{r=0}^{n} r^2 \binom{n}{r}.$$

B6. (a) Prove that, for all positive integers $n$ and all real numbers $x \neq 1$,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}.$$

By considering the integral between 0 and $\frac{1}{2}$ of $\sum_{r=0}^{\infty} x^r$, deduce that

$$\ln 2 = \sum_{r=1}^{\infty} \frac{1}{r2^r}.$$

(b) For $n$ a positive integer and $0 < t < \pi$ let

$$C_n(t) = \cos(t/2)\cos(t/4)\cdots\cos(t/2^n).$$

Show by induction on $n$ that

$$C_n(t) = \frac{\sin t}{2^n \sin(t/2^n)}.$$

## SECTION C

C7. Prove that

$$\int_0^a f(x)dx = \int_0^{\frac{a}{2}} [f(x) + f(a-x)]dx.$$

(i) Use this result to show that

$$\int_0^\pi x \sin^3 x \, dx = \pi \int_0^{\frac{\pi}{2}} \sin^3 x \, dx,$$

and hence evaluate $\int_0^\pi x \sin^3 x \, dx$.

(ii) Also use this result to show that,

$$\int_0^\pi \frac{x \sin x \, dx}{\sqrt{4-\cos^2 x}} = \frac{\pi^2}{6}.$$

C8. (a) Using the fact that, $\frac{d}{dx}\int_0^x g(t)dt = g(x)$, differentiate with respect to $x$ the following functions of $x$, when $0 \le x < \frac{\pi}{2}$.

(i) $\int_0^x \tan^{93} t \, dt$;

(ii) $\int_0^{x^2} \tan^{93} t \, dt$;

(iii) $\int_x^{x^2} \tan^{93} t \, dt$.

(b) Work out

$$\frac{d}{dx} \ln(\ln x) \text{ for } x > 1.$$

(Note that the function $\ln x$ can be defined as $\ln x = \int_1^x \frac{1}{t} dt$, for $x > 0$.) Using integration by parts evaluate

$$\int_2^x \frac{\ln(\ln t)}{t} dt, \text{ for } x \ge 2.$$

C9. Suppose that $y$ is a solution of the differential equation

$$(yx + 4x^2)\frac{dy}{dx} = 2y^2 + 9yx + 6x^2, \text{ for } x > 0.$$

By making the substitution $y = xu$ show that $u$ satisfies the differential equation

$$x\frac{du}{dx} = \frac{(u+3)(u+2)}{u+4}.$$

Integrate this equation and hence find a relation between $x$ and the solution $y$ of the original equation such that $y = 0$ when $x = 1$.

C10. Find the turning points on the graph of the function

$$y = -3x^3 + 9ax - 2a^2,$$

where $a$ is a real number. Sketch the curve (i) for $a < 0$, (ii) for $0 < a < 9$ and (iii) for $a > 9$. For each value of $a$ find the maximum value $\mathbf{M}(a)$ of the function in the interval $0 \leq x \leq 3$.

C11. Let $T_n(x) = \cos(n \cos^{-1} x)$, $\quad -1 \leq x \leq 1$, where $n$ is a nonnegative integer. (Note that another notation for $\cos^{-1} x$ is $\arccos x$.)

(a) Prove that, for $-1 < x < 1, T_n(x)$ satisfies the relation,

$$\frac{d}{dx}\left[\sqrt{1-x^2}\frac{d}{dx}(T_n(x))\right] = -\frac{n^2}{\sqrt{1-x^2}}T_n(x).$$

(b) Show that $T_n(x)$ has $n+1$ turning points in the interval $-1 \leq x \leq 1$.

(c) Using the substitution $x = \cos\theta$, or otherwise, prove that, if $m$ is a nonnegative integer such that $m \neq n$, then

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0.$$