CORE 2 MODULE REVISION SHEET

The C2 exam is 1 hour 30 minutes long and is in two sections.

Section A (36 marks) 8 – 10 short questions worth no more than 5 marks each.

Section B (36 marks) 3 questions worth 12 marks each.

You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

J M S

1. Sequences & Series

- A 'sequence' is a list of numbers in a specific order. A 'series' is a sum of the terms of a sequence.
- A periodic sequence repeats after some fixed number of terms (i.e. $a_{k+p} = a_k$ for some fixed value of p). For example $4, 1, 6, 7, 4, 1, 6, 7 \ldots$ is periodic.
- An oscillating sequence oscillates about some middle value. Therefore $3, 4, 5, 4, 3, 4, 5, 4, 3 \ldots$ is oscillating.
- Sequences are sometimes defined *inductively*. For example the sequence $a_{n+1} = a_n + 3$ with $a_1 = 10$ defines the sequence $10, 13, 16, 19 \ldots$ We know that this is an arithmetic sequence which can also be defined *deductively* by $a_n = 10 + 3(n - 1)$.

Arithmetic Sequences

- An arithmetic sequence increases or decreases by a constant amount. The letter a always denotes the first term and d is the difference between the terms (negative for a decreasing sequence!). The nth term is denoted a_n and satisfies the important relationship

$$a_n = a + (n - 1)d.$$

For example if told the third term of a sequence is 10 and the seventh term is 34 then we can use the above equation to find the a and d.

$$\begin{array}{l} 10 = a + (3-1)d \\ 34 = a + (7-1)d \end{array} \quad \Rightarrow \quad 4d = 24 \quad \Rightarrow \quad d = 6 \quad \Rightarrow \quad a = -2.$$

- The sum of the n terms of an arithmetic sequence is given by

$$S = \frac{n}{2}(\text{First} + \text{Last}) = \frac{n}{2}(2a + (n-1)d).$$

For example the sum of the first 10 terms of a sequence is 130 and the first term is 4. What is the difference?

$$S = \frac{n}{2}(2a + (n-1)d) \quad \Rightarrow \quad 130 = \frac{10}{2}(8 + (10-1)d) \quad \Rightarrow \quad d = 2.$$

Geometric Sequences

- A geometric sequence is one where the terms are multiplied by a constant amount. For example $1, 2, 4, 8, 16, \ldots, [2^{n-1}]$ is a geometric sequence with $a = 1$ and $r = 2$. The nth term is given by $a_n = ar^{n-1}$. So for the above example the 20th term is $a_{20} = 1 \times 2^{19} = 524288$.
- The sum of n terms of a geometric sequence is given by

$$S = a\frac{r^n - 1}{r - 1}.$$

For example sum the first 20 terms of $4, 2, 1, \frac{1}{2}, \ldots, [4 \times 2^{n-1}]$. This is given by $S = 4\frac{\left(\frac{1}{2}\right)^{20} - 1}{\frac{1}{2} - 1} = 7.999992371\ldots$

- If the ratio (r) is between -1 and 1 (i.e. $-1 < r < 1$) then there exists a 'sum to infinity' given by $S_\infty = \frac{a}{1-r}$. Therefore S_∞ for the above example is $S_\infty = \frac{4}{1-\frac{1}{2}} = 8$. We can see that the sum to 20 terms is very close to S_∞.

2. Differentiation

- Differentiation allows us to calculate the 'gradient function' $\dfrac{dy}{dx}$. This tells us how the gradient on the original function y changes with x.
- The rules are that;

$$y = \text{constant} \quad \Rightarrow \quad \frac{dy}{dx} = 0$$
$$y = ax \quad \Rightarrow \quad \frac{dy}{dx} = a$$
$$y = ax^n \quad \Rightarrow \quad \frac{dy}{dx} = anx^{n-1}$$

- For example $y = 4x^4 - 3x^2 + 2x - 5 \quad \Rightarrow \quad \dfrac{dy}{dx} = 16x^3 - 6x + 2$.
- Turning points are where the gradient of the curve is zero. They are either maxima, minima or points of inflection. To find the turning points of a curve we must find dy/dx and then set $dy/dx = 0$ and solve for x.
- To determine the nature of a turning point we must consider the sign of the gradient either side of the turning point. Present this in a table. In the example of $y = x^2 + 2x + 3$ we find $dy/dx = 2x + 2$ so we solve $0 = 2x + 2$ to give the turning point when $x = -1$:

x	$x < -1$	-1	$x > -1$
dy/dx	negative	0	positive
		minimum	

- We can also use the second derivative to determine the nature of a turning point. This is found by differentiating the function twice;

$$y = 2x^3 + 3x^2 - 2x + 4 \quad \Rightarrow \quad \frac{dy}{dx} = 6x^2 + 6x - 2 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 12x + 6.$$

You then evaluate the second derivative with the x value at the turning point and look at its sign. If it is positive it is a minimum, if it is negative it is a maximum. If it is zero then it is *probably* a point of inflection, but you need to do the above analysis either side of the turning point.

3. Integration

- Know that integration is the reverse of differentiation. That is if $\frac{dy}{dx} = f(x)$ then $y = \int f(x)\,dx$. For example if $\frac{dy}{dx} = 3x^3$ then $y = \int 3x^3\,dx = \frac{3}{4}x^4 + c$.
- The general rule is therefore $\displaystyle\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$.
- $\int y\,dx$ is an *indefinite* integral because there are no limits on the integral sign. When evaluating these integrals *never*[1] forget an *arbitrary constant* added on at the end. For example $\int 6x^2\,dx = 2x^3 + c$.
- $\int_a^b y\,dx$ is a *definite integral* and is the area between the curve and the x-axis from $x = a$ to $x = b$. Areas under the x-axis are negative. (For areas between the curve and the y-axis switch the x and the y and use $\int_p^q x\,dy$ between $y = p$ and $y = q$.)
- To find the area *between* two curves between $x = a$ and $x = b$ evaluate

$$\int_a^b (\text{top} - \text{bottom})\,dx.$$

- The area under *any* curve can be *approximated* by the Trapezium Rule. The governing formula is given by (and contained in the formula booklet you will have in the exam)

$$\int_a^b y\,dx \approx \frac{1}{2}h\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right],$$

  where h is the width of each trapezium, y_0 and y_n are the 'end' heights and $y_1 + y_2 + \cdots + y_{n-1}$ are the 'internal' heights.

4. Trigonometry

- The sine rule states for any triangle $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.
- The cos rule states that $a^2 = b^2 + c^2 - 2bc\cos A$. Practice both sine and cos rules on page 293.
- We define $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$. This identity is very useful in solving equations like $\sin\theta - 2\cos\theta = 0$ which yields $\tan\theta = 2$. The solutions of this in the range $0° \leqslant \theta \leqslant 360°$ are $\underline{\theta = 63.4°}$ and $\underline{\theta = 243.4°}$ to one decimal place.
- Know the following (or better yet, learn a couple and be able to quickly derive the rest from your knowledge of the trigonometric functions):

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	1	0	undefined
$180°$	0	-1	0

[1] *Never, never, never, never!* It's an easy mark on a 72 mark paper; don't lose it!

- Be able to sketch $\sin\theta$, $\cos\theta$ and $\tan\theta$ in both degrees and radians.

- By considering a right angled triangle (or a point on the unit circle) we can derive the important result $\sin^2\theta + \cos^2\theta \equiv 1$. This is useful in solving certain trigonometric equations. Worked example; solve $1 = 2\cos^2\theta + \sin\theta$ for $0° \leqslant \theta \leqslant 360°$.

$$\begin{aligned}
1 &= 2\cos^2\theta + \sin\theta \\
1 &= 2(1-\sin^2\theta) + \sin\theta && \text{get rid of } \cos^2\theta, \\
0 &= 1 - 2\sin^2\theta + \sin\theta && \text{quadratic in } \sin\theta, \\
0 &= 2\sin^2\theta - \sin\theta - 1 && \text{factorise as normal,} \\
0 &= (2\sin\theta + 1)(\sin\theta - 1).
\end{aligned}$$

  So we just solve $\sin\theta = -\frac{1}{2}$ and $\sin\theta = 1$. Therefore $\underline{\theta = 210°}$ or $\underline{\theta = 330°}$ or $\underline{\theta = 90°}$.

- By considering half of a general parallelogram we can show that the area of any triangle is given by $A = \frac{1}{2}ab\sin C$.

- There are (by definition) 2π radians in a circle. So $360° = 2\pi$. To convert from degrees to radians we use the conversion factor of $\frac{\pi}{180}$. For example to convert $45°$ to radians we calculate $45 \times \frac{\pi}{180} = \frac{\pi}{4}$ rad. From radians to degrees we use its reciprocal $\frac{180}{\pi}$.

- *When using radians* the formulae for arc length and area of a sector of a circle become simpler. They are $s = r\theta$ and $A = \frac{1}{2}r^2\theta$.

- Extending the results from C1 we find that given $y = f(x)$ then:

FUNCTION	GRAPH SHAPE
$f(x)$	Normal Graph
$2f(x)$	Graph stretched by a factor of 2 away from the x-axis i.e. every value of $f(x)$ in the original graph is multiplied by 2
$f(2x)$	Graph squeezed by factor of 2 towards the y-axis
$3f(4x)$	Graph squeezed by factor of 4 towards the y-axis followed by stretching by a factor of 3 away from the x-axis
$f(x)+6$	Graph moved vertically *up* 6 units
$f(x)-6$	Graph moved vertically *down* 6 units
$f(x+4)$	Graph moved 4 units to the *left*
$f(x-6)$	Graph moved 6 units to the *right*
$f(x-6)+9$	Graph translated 6 units to the *right* and 9 units *up*. This is a translation and can be expressed as $\binom{6}{9}$ where $\binom{\text{change in } x}{\text{change in } y}$
$-f(x)$	Graph reflected in the x-axis
$f(-x)$	Graph reflected in the y-axis

5. Logarithms & Exponentials

- Logarithms and exponentials (powers) are the inverse functions of each other. Therefore

$$\log_{10} 10^x = x \quad \text{and} \quad 10^{\log_{10} x} = x.$$

  So if $\log a = 5.4$ then $a = 10^{5.4}$.

- On page 325 there are lots of questions of the type ‘simplify $\log_4 2$’. In these types of questions you must write the number you’re logging as a power of the base of the logarithm. So in this case $\log_4 2 = \log_4 4^{\frac{1}{2}} = \frac{1}{2}\log_4 4 = \frac{1}{2}$.

- There are a few rules that you *must* learn about logarithms. They are (for all bases):

$$\log(ab) = \log a + \log b \qquad \log 1 = 0$$
$$\log(a/b) = \log a - \log b \qquad \log_a a = 1$$
$$\log(a^n) = n \log a \qquad \log_a b = \frac{\log_c b}{\log_c a}.$$
$$\log(1/a) = -\log a$$

- When we need to solve an equation where the unknown is in the exponent such as $5^{2x-1} = 8$ take $\log_{10}$ of both sides and simplify:

$$\begin{aligned} 5^{2x-1} &= 8 \\ \log_{10}(5^{2x-1}) &= \log_{10} 8 \\ (2x-1)\log_{10} 5 &= \log_{10} 8 \\ 2x-1 &= \frac{\log_{10} 8}{\log_{10} 5} \\ x &= \frac{1}{2} \times \left(\frac{\log_{10} 8}{\log_{10} 5} + 1\right) \\ x &= 1.15 \text{ (3sf)}. \end{aligned}$$

- There are two types of relationships that can be modelled by logarithms. They are *exponentials* of the form $y = ab^x$ and *polynomials* of the form $y = ax^b$. When one of these relationships is suggested in the exam, then take logs of both sides and rearrange to compare with $Y = mX + c$. Therefore

$$\begin{array}{cc} (A) & (B) \\ y = ab^x & y = ax^b \\ \log y = \log(ab^x) & \log y = \log(ax^b) \\ \log y = (\log b)x + \log a & \log y = b \log x + \log a. \end{array}$$

- So in (A) we plot x against $\log y$ and find a gradient of $\log b$ and y-axis intercept of $\log a$.
- In (B) we plot $\log x$ against $\log y$ and find a gradient of b and y-axis intercept of $\log a$. Do some examples of these questions on (and around) page 332.

6. Further Differentiation & Integration

- Nothing really new in this chapter, just extending the two important results for differentiation and integration to fractional indices.

  1. If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$.
  2. If $y = ax^n$ then $\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$.

- For example if $y = 4x^{\frac{5}{4}} + 3x^{\frac{4}{5}}$ then $\frac{dy}{dx} = 5x^{\frac{1}{4}} + \frac{12}{5}x^{-\frac{1}{5}}$.

- For example, given that $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ find the area under the curve from $x = 1$ and $x = 2$.

$$
\begin{aligned}
\int_1^2 \sqrt{x} + \frac{1}{\sqrt{x}}\, dx &= \int_1^2 x^{\frac{1}{2}} + x^{-\frac{1}{2}}\, dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_1^2 \\
&= \left(\frac{2}{3} \times 2^{\frac{3}{2}} + 2 \times 2^{\frac{1}{2}}\right) - \left(\frac{2}{3} + 2\right) \\
&= 2.05 \text{ (3sf)}.
\end{aligned}
$$