Integration

Between Curve & the x-axis

Most questions involve this type of integration!

> The area between a curve and the x-axis is given by $\int_a^b y\,dx$.

For example; "Find the area enclosed by the curve $y = x^2 + 4$, the x-axis and the lines $x = 2$ and $x = 5$." So we need to evaluate

$$\int_a^b y\,dx = \int_2^5 x^2 + 4\,dx = \left[\frac{x^3}{3} + 4x\right]_2^5 = \left(\frac{5^3}{3} + 4 \times 5\right) - \left(\frac{2^3}{3} + 4 \times 2\right) = 51$$

Between Curve & the y-axis

> The area between a curve and the y-axis is given by $\int_p^q x\,dy$.

For example; "Find the area enclosed by the curve $y = \sqrt{x+1}$, the y-axis and the (horizontal) lines $y = 3$ and $y = 7$." So we need to evaluate

$$\int_p^q x\,dy = \int_3^7 x\,dy.$$

Now the line is $y = \sqrt{x+1}$ so to find x in terms of y, we make x the subject;

$$y = \sqrt{x+1} \quad \Rightarrow \quad x = y^2 - 1.$$

We therefore evaluate

$$\int_p^q x\,dy = \int_3^7 y^2 - 1\,dy = \left[\frac{y^3}{3} - y\right]_3^7 = \left(\frac{7^3}{3} - 7\right) - \left(\frac{3^3}{3} - 3\right) = \frac{304}{3}$$

Volumes of Revolution

Around x-axis

> When rotating around the x-axis the volume of revolution is given by $\int_a^b \pi y^2\,dx$.

For example; "Find the volume of revolution of the solid formed by rotatating the curve $y = \sqrt{2x+3}$ about the x-axis between $x = 10$ and $x = 14$." So we need to evaluate

$$\int_a^b \pi y^2\,dx = \int_{10}^{14} \pi y^2\,dx.$$

Now the curve is $y = \sqrt{2x+3}$ so to find y^2 in terms of x, we need only square the equation $\Rightarrow y^2 = 2x + 3$. We therefore evaluate

$$\int_{10}^{14} \pi y^2\,dx = \pi \int_{10}^{14} (2x+3)\,dx = \pi\left[x^2 + 3x\right]_{10}^{14} = \pi[(14^2 + 3 \times 14) - (10^2 + 3 \times 10)] = 108\pi.$$

Around y-axis

When rotating around the y-axis the volume of revolution is given by $\int_p^q \pi x^2\, dy$.

For example; "Find the volume of revolution of the solid formed by rotatating the line $y = 3x-2$ about the y-axis between $y = 0$ and $y = 5$." So we need to evaluate

$$\int_p^q \pi x^2\, dy = \int_0^5 \pi x^2\, dy.$$

Now the line is $y = 3x - 2$ so to find x^2 in terms of y, we make x the subject and square;

$$y = 3x - 2 \quad \Rightarrow \quad x = \frac{y+2}{3} \quad \Rightarrow \quad x^2 = \frac{y^2+4y+4}{9}.$$

We therefore evaluate

$$\int_0^5 \pi x^2\, dy = \pi \int_0^5 \left(\frac{y^2+4y+4}{9}\right) dy = \frac{\pi}{9}\left[\frac{y^3}{3} + 2y^2 + 4y\right]_0^5 = \frac{\pi}{9}\left(\frac{5^3}{3} + 2\times 5^2 + 4\times 5\right) = \frac{335\pi}{27}.$$